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Research Article

The sensitivity analysis of population projections

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The sensitivity analysis of population projections

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Abstract

BACKGROUND
Population projections using the cohort component method can be written as time-varying matrix population models. The matrices are parameterized by schedules of mortality, fertility, immigration, and emigration over the duration of the projection. A variety of dependent variables are routinely calculated (the population vector, various weighted population sizes, dependency ratios, etc.) from such projections.

OBJECTIVE
Our goal is to derive and apply theory to compute the sensitivity and the elasticity (proportional sensitivity) of any projection outcome to changes in any of the parameters, where those changes are applied at any time during the projection interval.

METHODS
We use matrix calculus to derive a set of equations for the sensitivity and elasticity of any vector valued outcome $\boldsymbol{\xi}(t)$ at time t to any perturbation of a parameter vector $\boldsymbol{\theta}(s)$ at any time s.

RESULTS
The results appear in the form of a set of dynamic equations for the derivatives that are integrated in parallel with the dynamic equations for the projection itself. We show results for single-sex projections and for the more detailed case of projections including age distributions for both sexes. We apply the results to a projection of the population of Spain, from 2012 to 2052, prepared by the Instituto Nacional de Estadística, and determine the sensitivity and elasticity of (1) total population, (2) the school-age population, (3) the population subject to dementia, (4) the total dependency ratio, and (5) the economic support ratio.

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CONCLUSIONS
Writing population projections in matrix form makes sensitivity analysis possible. Such analyses are a powerful tool for the exploration of how detailed aspects of the projection output are determined by the mortality, fertility, and migration schedules that underlie the projection.

1. Introduction

Fifty years ago, in the first issue of the first volume of the then-new journal *Demography*, Nathan Keyfitz described the "population projection as a matrix operator" (Keyfitz 1964). He showed that population projections using the cohort component method could be written as matrix population models, and emphasized the value in doing so to focus attention on the mathematical structure of the projection, inviting deeper analyses of its properties with more powerful mathematical tools. Today, official projections are often implemented as computer algorithms, the details of which are obscure, but which permit almost endless fine-tuning of relationships. But the advantages of considering projections as matrix operators are no less real. In this paper, we carry on in this spirit, using matrix calculus methods to develop a thorough perturbation analysis of population projections.

As is customary in demography, we use the term *projection* to describe a conditional prediction of population size and structure, over a specified time horizon, such as are regularly developed by national governments, international consortia (e.g., Eurostat), and non-governmental organizations (U.N.). All projections are conditional in the sense that they are based on one or more hypothetical scenarios defining future rates of mortality, fertility, and migration (collectively, the "vital rates"), and also conditional on an initial population.

The vital rate scenarios are defined in terms of a set of parameters; the nature of those parameters will depend on the details of the scenarios. Sensitivity analysis (also called perturbation analysis) asks how the results of the projection would change in response to changes in the parameters. Sensitivity analysis is useful because:

1. It can project the consequences of changes in the vital rates. Such changes could result from human actions, either intentional (e.g., policies to encourage reproduction, public health interventions, or conservation strategies applied to endangered species) or unintentional (e.g., consequences of pollution or environmental degradation), or natural changes.
2. It can be used to compare potential policy interventions and identify interventions that would have particularly large effects. If an outcome is particularly sensitive to a particular parameter, that parameter may be an attractive target for intervention.

3. It can be used retrospectively to decompose observed changes in an outcome into contributions from changes in each of the parameters (Caswell 2000, 2001).
4. It can be used to identify parameters the estimation of which deserves extra attention, because they have large effects on the results.
5. It can quantify uncertainty of projection results: given the uncertainty in some parameter θ, and the sensitivity of an outcome of interest to changes in θ, it is possible to approximate the resulting uncertainty in the outcome. Demographers have become increasingly concerned with estimating the uncertainty of projection results (Booth 2006; Ahlburg and Lutz 1998).

In this paper, we focus on projections of populations classified by age and sex. Some projections are multistate models (e.g., projections of Belgium classify individuals by age, sex, nationality, and position in the household[3]; projections of Sweden classify individuals by age, sex, and country of birth[4]). We will present the sensitivity analysis of multistate projections in a subsequent paper.

1.1 Sensitivity and elasticity

Our approach is to calculate the derivatives of the projection results to the parameters and initial conditions. This gives the effects of small changes, gives approximate results for quite large changes, and identifies parameters with particularly large or small impacts on the results. As we will show, the parameters may include aspects of mortality, fertility, or immigration. The projection results may include a variety of different functions of the population, including measures of size, structure, and growth.

We will present results for both sensitivity and elasticity. If y is a function of x, we define the sensitivity of y to changes in x as

$$\text{sensitivity} = \frac{dy}{dx}. \tag{1}$$

The elasticity of y is the proportional sensitivity, which is

$$\begin{aligned} \text{elasticity} &= \frac{x}{y}\frac{dy}{dx} && (2) \\ &= \frac{\epsilon y}{\epsilon x}. && (3) \end{aligned}$$

This gives the proportional change in y resulting from a proportional change in x. There is no standard notation for elasticities, despite their widespread use in economics and population biology. The notation used here, $\epsilon y / \epsilon x$, which parallels the notation for derivatives,

[3] http://statbel.fgov.be/fr/binaries/FORPOP1360_10697_F_tcm326-244744.pdf
[4] http://www.scb.se/statistik/_publikationer/BE0401_2012I60_BR_BE51BR1202ENG.pdf

is adapted from a notation used by Samuelson (1947). Elasticities are defined only when $y > 0$ and $x \geq 0$.

In Section 2 we will write both one-sex and two-sex projections as matrix operators, and discuss the scenarios that might be involved in such projections and the parameters that might determine those scenarios. In Section 3 we will give the expressions for the sensitivities and elasticities of the population vector (abundance by age class of males, or females, or both combined) to changes in mortality, fertility, and immigration. A particularly important part of our results, in Section 3.5, is to show how the sensitivity results for the population vector can be translated directly into other dependent variables, such as weighted population size, ratios, and growth rates. The derivations of results are given in detail in Appendix A.

Our approach is to write the projection as a matrix operator, and then to use matrix calculus (e.g., Caswell 2007, 2008, 2009; Caswell and Shyu 2012) to derive the needed derivatives of the results to underlying parameters. These methods are easily implemented in any matrix-oriented computer language, especially MATLAB, but also R.

In Section 4 we will apply the calculations to a projection of the population of Spain, using information from the Instituto Nacional de Estadística (INE). We conclude with a discussion of how these results apply to evaluating the uncertainty of projections and future developments.

Notation. Matrices are denoted by upper case bold symbols (e.g., $\mathbf{A}$) and vectors by lower case bold symbols (e.g., $\mathbf{n}$). All vectors are column vectors by default. The vector $\mathbf{x}^{\mathsf{T}}$ is the transpose of the vector $\mathbf{x}$. The Hadamard, or element-by-element, product of $\mathbf{A}$ and $\mathbf{B}$ is $\mathbf{A} \circ \mathbf{B}$. The Kronecker product is $\mathbf{A} \otimes \mathbf{B}$. The diagonalization operator $\mathrm{diag}\,(\mathbf{x})$ creates a matrix with $\mathbf{x}$ on the diagonal and zeros elsewhere. The vec operator, when applied to a $m \times n$ matrix $\mathbf{X}$ creates a $mn \times 1$ vector $\mathrm{vec}\,\mathbf{X}$ by stacking each column of $\mathbf{X}$ on top of the next. The vector $\mathbf{1}$ is a vector of ones, and the vector $\mathbf{e}_i$ is the ith unit vector, with a 1 in the ith location and zeros elsewhere. When necessary, subscripts are attached to indicate the size of matrices or vectors; e.g., $\mathbf{I}_\omega$ is the $\omega \times \omega$ identity matrix.

2. Projection as a matrix operation

2.1 Dynamics

To present the basics of projection sensitivity analysis, we begin with a simple one-sex model, but we focus most of our attention on a two-sex model that includes separate rates for males and females.

The single-sex projection can be written as

$$\mathbf{n}(t+1) = \mathbf{A}(t)\mathbf{n}(t) + \mathbf{b}(t) \qquad \mathbf{n}(0) = \mathbf{n}_0 \tag{4}$$

where $\mathbf{n}(t)$ is a vector whose entries are the numbers of individuals in each age class or stage at time t, $\mathbf{A}(t)$ is a projection matrix incorporating the vital rates at time t, and $\mathbf{b}(t)$ is a vector giving the number of immigrants in each age class or stage at time t. The projection begins with a specified initial condition, denoted $\mathbf{n}_0$, and is carried out until some target time T.

Two-sex projections are generalizations of (4). We define population vectors $\mathbf{n}_f$ and $\mathbf{n}_m$, and projection matrices $\mathbf{A}_f$ and $\mathbf{A}_m$, for females and males, respectively. We assume that reproduction is female dominant[5], so all fertility is attributed to females. We decompose the projection matrices for females and males into

$$\mathbf{A}_f(t) = \mathbf{U}_f(t) + \phi\mathbf{F}(t) \tag{5}$$

$$\mathbf{A}_m(t) = \mathbf{U}_m(t) \tag{6}$$

where $\mathbf{U}$ describes transitions and survival of extant individuals and $\mathbf{F}$ describes the production of new individuals by reproduction.

In an age-classified model, $\mathbf{F}$ will have effective fertilities (including infant and maternal survival as appropriate) on the first row and zeros elsewhere. A proportion ϕ of the offspring are **female**. This model attributes reproduction to females; hence there is no need to create separate fertility matrices for reproduction by males and females.

The male component of the population is projected by the survival matrix $\mathbf{U}_m$; the input of new individuals comes from the female population. The projection model becomes

$$\mathbf{n}_f(t+1) = \Big[\mathbf{U}_f(t) + \phi\mathbf{F}(t)\Big]\,\mathbf{n}_f(t) + \mathbf{b}_f(t) \tag{7}$$

$$\mathbf{n}_m(t+1) = \mathbf{U}_m(t)\mathbf{n}_m(t) + (1-\phi)\mathbf{F}(t)\mathbf{n}_f(t) + \mathbf{b}_m(t). \tag{8}$$

The formulations in equations (4), (7), and (8) are general enough to encompass all the projections typically used. The vector $\mathbf{n}$ can incorporate any type of population structure considered relevant. If individuals are grouped into age classes, then $\mathbf{A}$ is the familiar Leslie matrix, with survival probabilities on the subdiagonal, fertilities in the first row, and zeros elsewhere. If individuals are classified by other criteria ("stages" in common usage), $\mathbf{A}$ will have the structure needed to capture transitions among stages based on physiological condition, developmental stage, socio-economic grouping, marital status, parity status, etc.

Immigration, denoted here by $\mathbf{b}(t)$, is a particularly challenging part of population projection. We explore the reasons for this, and some of the ways in which migration is handled, in Section 6.3. Some implementations of migration require minor modifications

[5] Two-sex models that do not assume dominance by one sex have been used to project animal populations (e.g., Jenouvrier et al. 2009, 2012, 2014), but rarely, as far as we know, human populations (but see Ekamper and Keilman (1993)).

of equations (4)–(8), but the sensitivities are derived in the same way as what we are about to show.

2.2 Scenarios and parameters

A projection is based on a scenario of how the future might unfold. The matrices $\mathbf{U}(t)$ and $\mathbf{F}(t)$, and the vector $\mathbf{b}(t)$, describe the future dynamics of mortality, fertility, and immigration. The future being unknown, considerable ingenuity is required to construct these functions. Three major approaches seem to be used, singly or in combination.

1. Extrapolation of trends. This approach starts from the observation that some vital rates (particularly mortality and fertility rates) develop gradually over time, and extrapolates those patterns into the future. The best-known of these is perhaps the Lee-Carter model for mortality, which projects mortality with a time-series model applied to a singular value decomposition of a past record of age- and time-specific mortality rates. Recent developments include sophisticated Bayesian methods that also produce statistically rigorous uncertainty bounds (e.g., Gerland et al. 2014).

2. Assumptions and expert opinion. Future trends in vital rates are sometimes simply assumed, based on unspecified conceptual models. The projections of Eurozone countries by Eurostat, for example, are based on the assumption that the mortality and fertility of all European countries will converge to a common value by the year 2150 (Lanzieri 2009). The rates for a given country in each year are determined by interpolating between the rates at the start of the projection and the final target rates. Other studies have been based on the opinion of experts who are not directly involved in the projection process. Lutz and colleagues, for instance, have used a Delphi-method based approach to collect and aggregate external expert opinions on demographic trends in a systematic manner (Ahlburg and Lutz 1998). Expectations of population members about their own lives (e.g. survey data on the expected number of children or expected remaining life expectancy) have also been used to define scenarios.

3. Dependence on external factors, which can themselves be projected. If the vital rates depend on some factor, and the dynamics of that factor can be predicted, this provides the basis for a projection of the vital rates. The approach has been used for animal populations. For example, projections of populations of polar bears and emperor penguins under the impact of climate change have been based on projections of sea ice conditions (a critical environmental variable for these species) generated by models of global climate conditions produced by the IPCC (Hunter et al. 2010;

Jenouvrier et al. 2009, 2012, 2014). Similarly, projections of human populations have been based on expectations about future economic, social or environmental developments (Booth 2006).

Regardless of how the scenario of future conditions is obtained, the resulting projection depends on a set of *parameters* which jointly determine the projection matrices and the immigration vectors. We will write this set of parameters as a vector $\boldsymbol{\theta}$, of dimension p. In this paper, we focus on the commonly encountered case in which the parameters are the age- and time-specific rates of mortality, fertility, and immigration:

$$\boldsymbol{\theta}(t) = \begin{cases} \boldsymbol{\mu}(t) & \text{vector of mortality rates} \\ \mathbf{f}(t) & \text{vector of age-specific fertility} \\ \mathbf{b}(t) & \text{immigration vector} \end{cases} \tag{9}$$

These vectors might, in turn, be expressed as functions of a scalar quantity such as life expectancy, or a parametric model such as the Gompertz, gamma-Gompertz, or Siler models for mortality, or the Coale-Trussell function for fertility. In that case, the vector $\boldsymbol{\theta}$ would include the parameters that define those functions.

3. Perturbation analysis of projections

Our goal is to quantify the sensitivity and elasticity of projection results to the parameters in $\boldsymbol{\theta}$. To do that, we need to introduce the matrix calculus framework for derivatives of vectors (the projection output) with respect to other vectors (the parameter vector). The derivations of our results are given in detail in Appendix A.

3.1 Matrix calculus notation

Matrix calculus permits the differentiation of scalar-, vector-, or matrix-valued functions of scalar-, vector-, or matrix-valued arguments. The underlying theory is presented in detail by Magnus and Neudecker (1985); for an introductory account see Abadir and Magnus (2005). The methods have been applied to demography in a series of papers (Caswell 2006, 2007, 2008, 2010, 2012; Caswell and Shyu 2012; van Raalte and Caswell 2013; Engelman, Caswell, and Agree 2014).

If $\mathbf{y}$ is a $n \times 1$ vector function of the $m \times 1$ vector $\mathbf{x}$, then the sensitivity of $\mathbf{y}$ to $\mathbf{x}$ is the $n \times m$ Jacobian matrix written as

$$\frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}} = \left(\frac{\partial y_i}{\partial x_j} \right) \qquad i = 1, \ldots, m;\ j = 1, \ldots, n. \tag{10}$$

We will use the fact that this calculus satisfies the chain rule, so that if $\mathbf{z}$ is a function of $\mathbf{y}$, then

$$\frac{d\mathbf{z}}{d\mathbf{x}^{\mathsf{T}}} = \frac{d\mathbf{z}}{d\mathbf{y}^{\mathsf{T}}} \frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}}. \tag{11}$$

The elasticity, or proportional sensitivity, of $\mathbf{y}$ with respect to $\mathbf{x}$ is the $n \times m$ matrix given by

$$\frac{\epsilon \mathbf{y}}{\epsilon \mathbf{x}^{\mathsf{T}}} = \text{diag}\,(\mathbf{y})^{-1} \left(\frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}} \right) \text{diag}\,(\mathbf{x}). \tag{12}$$

We will present a series of sensitivity and elasticity relationships of the form

$$\frac{d\boldsymbol{\xi}}{d\boldsymbol{\theta}^{\mathsf{T}}} \quad \text{and} \quad \frac{\epsilon \boldsymbol{\xi}}{\epsilon \boldsymbol{\theta}^{\mathsf{T}}}$$

where $\boldsymbol{\xi}$ is a projection output and $\boldsymbol{\theta}$ is a vector of parameters. The output $\boldsymbol{\xi}$ might be the population vector $\mathbf{n}(t)$, or it might be some function of $\mathbf{n}$ (e.g., a dependency ratio). The sensitivity of $\boldsymbol{\xi}$ is obtained from a system of equations giving a dynamic model (i.e. a model specifying changes through time) for the derivatives of the population vector at time t to a parameter change at time s

$$\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}$$

where s is the time at which the parameter vector $\boldsymbol{\theta}$ is perturbed. If there are ω age classes and p parameters, then this derivative is a $\omega \times p$ matrix whose (i, j) entry is the derivative of $n_i(t)$ with respect to the parameter θ_j at time s.

3.2 One-sex projections

For simplicity, we begin with the one-sex projection (4). We consider the effects of changes in the parameters at time s on the projected population at time t, for $s = 0, \ldots, T$ and $t = s, \ldots, T$. Changes in $\boldsymbol{\theta}(s)$ obviously have no effect on $\mathbf{n}(t)$ at an earlier time $t < s$ (we ignore the complications of time travel). However, a perturbation at time s will ripple through $\mathbf{n}(t)$ for all later times $t > s$, and our goal is to find out how.

The dynamics of the population vector $\mathbf{n}(t)$ are obtained by iterating equation (4). The sensitivity of $\mathbf{n}(t)$ to a change in $\boldsymbol{\theta}(s)$ is obtained by iterating the dynamic equation

$$\frac{d\mathbf{n}(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \mathbf{A}(t) \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + \left(\mathbf{n}^{\mathsf{T}}(t) \otimes \mathbf{I}_{\omega} \right) \frac{d\text{vec}\,\mathbf{A}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + \frac{d\mathbf{b}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \tag{13}$$

starting from the initial condition

$$\frac{d\mathbf{n}(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \mathbf{0}_{\omega \times p}. \tag{14}$$

The elasticity of $\mathbf{n}(t)$ to $\boldsymbol{\theta}(s)$ is, from (12),

$$\frac{\epsilon \mathbf{n}(t)}{\epsilon \boldsymbol{\theta}^{\mathsf{T}}(s)} = \operatorname{diag}\left[\mathbf{n}(t)\right]^{-1} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \operatorname{diag}\left[\boldsymbol{\theta}(s)\right]. \tag{15}$$

The structure of (13) is common to all the sensitivity results; it contains terms involving the sensitivity at time t and terms that update that sensitivity with effects on $\mathbf{A}(t)$ and $\mathbf{b}(t)$, giving the sensitivity at $t+1$:

$$\underbrace{\frac{d\mathbf{n}(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}}_{\text{sensitivity at } t+1} = \underbrace{\mathbf{A}(t)\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}}_{\text{sensitivity at } t} + \underbrace{\left(\mathbf{n}^{\mathsf{T}}(t) \otimes \mathbf{I}_{\omega}\right) \frac{d\operatorname{vec}\mathbf{A}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}}_{\text{effects via } \mathbf{A}} + \underbrace{\frac{d\mathbf{b}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}}_{\text{effects via } \mathbf{b}}. \tag{16}$$

3.3 Two-sex projections

The sensitivity of the two-sex projection is given by the two derivatives,

$$\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \quad \text{and} \quad \frac{d\mathbf{n}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}.$$

These derivatives are obtained from dynamic expressions, for the female population

$$\begin{aligned}
\frac{d\mathbf{n}_f(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} &= \Big(\mathbf{U}_f(t) + \phi\mathbf{F}(t)\Big)\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \\
&\quad + \left(\mathbf{n}_f^{\mathsf{T}}(t) \otimes \mathbf{I}_{\omega}\right)\left(\frac{d\operatorname{vec}\mathbf{U}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + \phi\frac{d\operatorname{vec}\mathbf{F}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}\right) + \frac{d\mathbf{b}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}
\end{aligned} \tag{17}$$

and the male population

$$\begin{aligned}
\underbrace{\frac{d\mathbf{n}_m(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}}_{\text{sensitivity at } t+1} &= \underbrace{\mathbf{U}_m(t)\frac{d\mathbf{n}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + (1-\phi)\mathbf{F}(t)\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}}_{\text{sensitivities at } t} \\
&\quad + \underbrace{\left(\mathbf{n}_m^{\mathsf{T}}(t) \otimes \mathbf{I}_{\omega}\right)\frac{d\operatorname{vec}\mathbf{U}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}}_{\text{effects via male transitions}} + \underbrace{(1-\phi)\left(\mathbf{n}_f^{\mathsf{T}}(t) \otimes \mathbf{I}_{\omega}\right)\frac{d\operatorname{vec}\mathbf{F}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}}_{\text{effects via female fertility}} \\
&\quad + \underbrace{\frac{d\mathbf{b}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}}_{\text{effects via immigration}}.
\end{aligned} \tag{18}$$

Equations (17) and (18) are iterated from initial conditions

$$\frac{d\mathbf{n}_f(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \frac{d\mathbf{n}_m(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \mathbf{0}_{\omega\times p} \tag{19}$$

along with the iteration of equations (7) and (8) for the population vectors $\mathbf{n}_f(t)$ and $\mathbf{n}_m(t)$.

We have labelled the terms in (18) to show the parallels with the simpler one-sex model (16). Again, the sensitivity at $t+1$ depends on the sensitivity at time t and on the effects of the parameter vector on the transition and fertility matrices and on the immigration vector. In the next section we turn to the calculation of these derivatives.

The elasticities of $\mathbf{n}_f(t)$ and $\mathbf{n}_m(t)$ are given by applying (15) to the corresponding derivatives for female and male population:

$$\frac{\epsilon\mathbf{n}_f(t)}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(s)} = \text{diag}\left[\mathbf{n}_f(t)\right]^{-1}\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}\,\text{diag}\left[\boldsymbol{\theta}(s)\right] \tag{20}$$

and similarly for $\mathbf{n}_m$.

The combined population of both males and females is $\mathbf{n}_c = \mathbf{n}_f + \mathbf{n}_m$. The sensitivity and elasticity of $\mathbf{n}_c$ are

$$\frac{d\mathbf{n}_c(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + \frac{d\mathbf{n}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \tag{21}$$

$$\frac{\epsilon\mathbf{n}_c(t)}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(s)} = \text{diag}\left[\mathbf{n}_c(t)\right]^{-1}\left[\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + \frac{d\mathbf{n}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}\right]\text{diag}\left[\boldsymbol{\theta}(s)\right]. \tag{22}$$

The entire system of sensitivity and elasticity relationships is obtained by simultaneously iterating equations (7) and (8) to project the populations of females and males, and the equations (17) and (18) to project the sensitivity of the female and male populations.

3.4 Parameters and the derivatives of matrices

So far we have left the parameter vector $\boldsymbol{\theta}$ undefined, because the results apply to any choice of parameter. Now we become more specific by focusing on the cases where $\boldsymbol{\theta}$ is a vector of mortality rates, or of fertilities, or of immigration numbers. We consider each of these cases and present the derivatives of the matrices $\mathbf{U}$ and $\mathbf{F}$, and the vector $\mathbf{b}$, to those parameters. These derivatives appear in the expressions (17), (18), and (21) and the corresponding elasticity equations.

A change in the parameter vector $\boldsymbol{\theta}$ at time s can affect the projection matrices only when $t = s$; to indicate this, we will use the Kronecker delta function

$$\delta(s,t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases} \tag{23}$$

Because sex-specific mortality affects the matrices for only that sex, the following results apply to either male or female rates, so we do not include the subscript that defines the sex of the subpopulation.

- **Mortality:** $\boldsymbol{\theta} = \boldsymbol{\mu}$. Mortality rates affect the transition matrix $\mathbf{U}$ (or the projection matrix $\mathbf{A}$, if transitions and fertility are not separated). Define the survival vector $\mathbf{p} = \exp(-\boldsymbol{\mu})$, which appears on the subdiagonal of $\mathbf{U}$, and an indicator matrix $\mathbf{Z}$ with ones on the subdiagonal and zeros elsewhere. Then

$$\frac{d\mathrm{vec}\,\mathbf{A}(t)}{d\boldsymbol{\mu}^{\mathsf{T}}(s)} = \frac{d\mathrm{vec}\,\mathbf{U}(t)}{d\boldsymbol{\mu}^{\mathsf{T}}(s)} = -\delta(s,t)\mathrm{diag}\left(\mathrm{vec}\,\mathbf{Z}\right)\left(\mathbf{I}_\omega \otimes \mathbf{1}_\omega\right)\mathrm{diag}\,\big(\mathbf{p}(t)\big) \tag{24}$$

  where $\mathbf{1}$ is a vector of ones. The derivatives of $\mathbf{F}$ and $\mathbf{b}$ with respect to $\boldsymbol{\mu}$ are zero.

- **Fertility:** $\boldsymbol{\theta} = \mathbf{f}$. The fertility vector appears on the first row of the matrix $\mathbf{F}$. The derivative of $\mathbf{F}$ is

$$\frac{d\mathrm{vec}\,\mathbf{F}(t)}{d\mathbf{f}^{\mathsf{T}}(s)} = \delta(s,t)\left(\mathbf{I}_\omega \otimes \mathbf{e}_1\right) \tag{25}$$

  where $\mathbf{e}_1$ is the first unit vector, of length ω. The derivatives of $\mathbf{U}$ and $\mathbf{b}$ with respect to $\mathbf{f}$ are zero.

- **Immigration:** $\boldsymbol{\theta} = \mathbf{b}$. When the parameter vector is the immigration vector, then

$$\frac{d\mathbf{b}(t)}{d\mathbf{b}^{\mathsf{T}}(s)} = \delta(s,t)\mathbf{I}_\omega \tag{26}$$

  and the derivatives of $\mathbf{U}$, $\mathbf{F}$, and $\mathbf{A}$ with respect to $\mathbf{b}$ are all zero.

- **Initial population:** $\boldsymbol{\theta} = \mathbf{n}_0$. It is also possible to calculate the sensitivity of $\mathbf{n}(t)$ to the initial population vector. The derivatives of $\mathbf{U}$, $\mathbf{F}$, $\mathbf{A}$, and $\mathbf{b}$ with respect to the initial population are all zero, and the derivatives of the population vector reduce to an especially simple form.

  Let $\mathbf{n}_{0,f}$ and $\mathbf{n}_{0,m}$ be the initial male and female population vectors. The female population is independent of the initial male population, but the male population will depend on both the male and female initial populations. From equations (17) and (18) we have for the female population:

$$\frac{d\mathbf{n}_f(t+1)}{d\mathbf{n}_{0,f}^{\mathsf{T}}} = \left(\mathbf{U}_f(t) + \phi\mathbf{F}(t)\right)\frac{d\mathbf{n}_f(t+1)}{d\mathbf{n}_{0,f}^{\mathsf{T}}} \tag{27}$$

  with initial condition

$$\frac{d\mathbf{n}_f(0)}{d\mathbf{n}_{0,f}^{\mathsf{T}}} = \mathbf{I}_\omega. \tag{28}$$

For the male population, we have dependence on the initial male population

$$\frac{d\mathbf{n}_m(t+1)}{d\mathbf{n}_{0,m}^{\mathsf{T}}} = \mathbf{U}_m(t)\frac{d\mathbf{n}_m(t)}{d\mathbf{n}_{0,m}^{\mathsf{T}}} \tag{29}$$

with initial condition

$$\frac{d\mathbf{n}_m(0)}{d\mathbf{n}_{0,m}^{\mathsf{T}}} = \mathbf{I}_\omega. \tag{30}$$

The dependence on the initial female population we have

$$\frac{d\mathbf{n}_m(t+1)}{d\mathbf{n}_{0,f}^{\mathsf{T}}} = \mathbf{U}_m(t)\frac{d\mathbf{n}_m(t)}{d\mathbf{n}_{0,f}^{\mathsf{T}}} + (1-\phi)\mathbf{F}(t)\frac{d\mathbf{n}_f(t)}{d\mathbf{n}_{0,f}^{\mathsf{T}}} \tag{31}$$

with initial condition

$$\frac{d\mathbf{n}_m(0)}{d\mathbf{n}_{0,f}^{\mathsf{T}}} = \mathbf{0}_{\omega\times\omega}. \tag{32}$$

The behavior of these sensitivities depends on the behavior of the population and the cohorts that comprise it. From (27), it is apparent that the sensitivity of the female population to the initial female population grows or shrinks depending on whether the sequence of projection matrices, $(\mathbf{U}_f(t) + r\mathbf{F}(t))$, tends to expand or contract the population. The sensitivity of the male population to the initial male population decays according to the sequence of survival matrices $\mathbf{U}(t)$. Thus, in general, the sensitivity to initial population will play a greater role in rapidly expanding populations (a fact easily predictable from general principles of linear system theory). In populations that grow only slightly, or decline, over the time span of the projection, these sensitivities will be less important.

3.5 Choosing a dependent variable

These results in equations (16), (17), and (18) provide the sensitivity of *every* age class, at *every* time from 0 to T, with respect to changes in mortality, fertility, and immigration of *every* age class, at *every* time from 0 to T. This high-dimensional data structure is more information than anyone wants. It must be condensed to provide the sensitivity of specific projection outcomes of interest. An informal survey of Statistical Offices[6] finds that they typically present projections of (1) the total population size, (2) the proportional representation of specific age groups (e.g., working-age adults, school-age children), (3) ratios such as the old-age, young-age, and total dependency ratios, (4) and descriptors of

[6]European Union, Germany, France, Belgium, Ireland, Estonia, Spain, Austria, Sweden, United Kingdom, Iceland, and Switzerland

the age distribution such as the mean age in the population. Sometimes projections also include results on the population subject to particular diseases or handicaps.

In this section, we show how to calculate the sensitivity and elasticity of such dependent variables from the derivatives of $\mathbf{n}(t)$ given in (17), (18), and (21). In the following, sensitivities can be applied to the female population, the male population, or the combined population. Derivations are given in Appendix A.3.

1. Total population size $N(t)$. The total population size is $N(t) = \mathbf{1}_\omega^\mathsf{T}\mathbf{n}(t)$; its sensitivity to parameter changes at time s is

$$\frac{dN(t)}{d\boldsymbol{\theta}^\mathsf{T}(s)} = \mathbf{1}_\omega^\mathsf{T}\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^\mathsf{T}(s)}. \tag{33}$$

The elasticity of $N(t)$ is

$$\frac{\epsilon N(t)}{\epsilon\boldsymbol{\theta}^\mathsf{T}(s)} = \frac{1}{N(t)}\frac{dN(t)}{d\boldsymbol{\theta}^\mathsf{T}(s)}\ \mathrm{diag}\,(\boldsymbol{\theta}). \tag{34}$$

2. *Weighted* total population size. Define $N(t) = \mathbf{c}^\mathsf{T}\mathbf{n}(t)$, where $\mathbf{c}$ is a vector that applies different weights to each age class. For example, $\mathbf{c}$ might contain the labor income of each age class, or the prevalence in each age class of some health condition. $N(t)$ is now a weighted population size; the sensitivity of $N(t)$ to a change in parameters at time s is

$$\frac{dN(t)}{d\boldsymbol{\theta}^\mathsf{T}(s)} = \mathbf{c}^\mathsf{T}\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^\mathsf{T}(s)}. \tag{35}$$

The elasticity is again given by (34). If the weight vector $\mathbf{c}$ is subject to perturbations (e.g., if the prevalence of a health condition were to change due to screening or treatment), the sensitivity of $N(t)$ to changes in $\mathbf{c}$ is

$$\frac{dN(t)}{d\mathbf{c}^\mathsf{T}} = \mathbf{n}^\mathsf{T}(t). \tag{36}$$

The corresponding elasticities of $N(t)$ to $\mathbf{c}$ are

$$\frac{\epsilon N(t)}{\epsilon\mathbf{c}^\mathsf{T}} = \frac{1}{N(t)}\mathbf{n}^\mathsf{T}(t)\mathrm{diag}\,(\mathbf{c}). \tag{37}$$

The elasticities of $N(t)$ to $\mathbf{c}$ in (37) always sum to 1.

3. Ratios of weighted population sizes. Let

$$R(t) = \frac{\mathbf{a}^{\mathsf{T}}\mathbf{n}(t)}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)}, \tag{38}$$

where $\mathbf{a}$ and $\mathbf{c}$ are vectors of weights. The sensitivity of such a ratio (Caswell 2007) is

$$\frac{dR(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \left(\frac{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\mathbf{a}^{\mathsf{T}} - \mathbf{a}^{\mathsf{T}}\mathbf{n}(t)\mathbf{c}^{\mathsf{T}}}{\left[\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\right]^2} \right) \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}. \tag{39}$$

The elasticity of $R(t)$ is

$$\frac{\epsilon R(t)}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(s)} = \frac{1}{R(t)} \frac{dR(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \,\mathrm{diag}\,\left[\boldsymbol{\theta}(s)\right]. \tag{40}$$

Such ratios appear frequently as dependent variables in population projections. Examples include:

(a) The proportional representation of an age group (e.g., the proportion over 65 years of age). In this case, $\mathbf{a}$ is an indicator vector, containing ones corresponding to the ages in the age group, and zeros elsewhere. The vector $\mathbf{c} = \mathbf{1}$, so that $\mathbf{c}^{\mathsf{T}}\mathbf{N}$ is the total population size.

(b) Dependency ratios. In this case, $\mathbf{a}$ and $\mathbf{c}$ are both indicator vectors for the relevant age groups. The old-age dependency ratio, for example, is obtained by letting $\mathbf{a}$ indicate ages beyond retirement age and $\mathbf{c}$ indicate working ages.

(c) Weighted dependency ratios. Instead of considering all individuals of retirement age, or working age, to be equal, $\mathbf{a}$ and $\mathbf{c}$ can be vectors of weights. For example, the economic support ratio (Prskawetz and Sambt 2014) is computed by letting $\mathbf{a}$ be a vector giving age-specific labor income, and $\mathbf{c}$ a vector giving age-specific consumption.

(d) Moments of the age distribution. The mean of the age distribution is obtained by setting the vector $\mathbf{a}$ to the midpoints of the age intervals; e.g., for one year age classes,

$$\mathbf{a} = \begin{pmatrix} 0.5 & 1.5 & 2.5 & \cdots \end{pmatrix}^{\mathsf{T}} \tag{41}$$

and setting $\mathbf{c} = \mathbf{1}$. The second moment of the age distribution is obtained by setting

$$\mathbf{a} = \begin{pmatrix} 0.5^2 & 1.5^2 & 2.5^2 & \cdots \end{pmatrix}^{\mathsf{T}} \tag{42}$$

and $\mathbf{c} = 1$. The variance in age is obtained from the first and second moments in the usual way.

(e) Moments of age-specific properties. Suppose that $B(s)$ is some measurement on age class x (e.g., the mean body mass index (BMI) of age class x). Then the mean BMI in the population would be obtained by setting $\mathbf{c} = \mathbf{1}$ and

$$\mathbf{a} = \begin{pmatrix} B(1) & B(2) & B(3) & \cdots \end{pmatrix}^{\mathsf{T}}. \tag{43}$$

4. Short-term growth rates. Define the k-step growth rate of the weighted population size $\mathbf{c}^{\mathsf{T}}\mathbf{n}$, at time t as

$$\lambda(t) = \frac{\boldsymbol{c}^{\mathsf{T}}\mathbf{n}(t+k)}{\boldsymbol{c}^{\mathsf{T}}\mathbf{n}(t)}. \tag{44}$$

This gives the average discrete-time growth rate of the population over the next k years, starting from year t. The sensitivity of $\lambda(t)$ is

$$\frac{d\lambda(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \frac{\mathbf{c}^{\mathsf{T}}}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} \frac{d\mathbf{n}(t+k)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} - \frac{\lambda(t)\mathbf{c}^{\mathsf{T}}}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}. \tag{45}$$

The quantity λ is a discrete time growth rate; the sensitivity of the corresponding continuous time, annual growth rate $r(t) = \log \lambda(t)/k$ is

$$\frac{dr(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \frac{1}{k\lambda(t)} \frac{d\lambda(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}. \tag{46}$$

3.6 Aggregating perturbations over age and time

The expressions presented so far give the response of the dependent variable at any time t, to a perturbation of any of the parameters in $\boldsymbol{\theta}$, at any other time s. It is often useful to aggregate sensitivity over age, or over time, or over parameters, or all of these. Some examples are:

1. The sensitivity of $\mathbf{n}$ at time t to a perturbation, at time s, that affects the mortality, fertility, or immigration of all age classes by the same amount. The sensitivity of $\mathbf{n}(t)$ to an additive perturbation at every age is the sum of the sensitivities to the age-specific rates. The elasticity of $\mathbf{n}(t)$ to a proportional perturbation at every age is the sum of the elasticities to the age-specific rates. Thus, whatever rates $\boldsymbol{\theta}(s)$ may refer to, the sensitivity and elasticity are given by

$$\text{sensitivity:} \qquad \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \mathbf{1}_p \tag{47}$$

$$\text{elasticity:} \qquad \frac{\epsilon\mathbf{n}(t)}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(s)} \mathbf{1}_p. \tag{48}$$

2. The sensitivity of the population vector at time t to a change in $\boldsymbol{\theta}(s)$ that is applied equally at every time from $s = 0$ to $s = T$. A possible example of interest might be the sensitivity of the population vector in the final projection year, $\mathbf{n}(T)$, to a change in fertility at every age or at a specific age group over the entire projection period. In a slight abuse of notation, let us denote the sensitivity of $\mathbf{n}(T)$ to this type of perturbation as

$$\frac{d\mathbf{n}(T)}{d\boldsymbol{\theta}^{\mathsf{T}}(0,T)} = \sum_{s=0}^{T} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}. \tag{49}$$

The corresponding elasticity is

$$\begin{aligned} \frac{\epsilon\mathbf{n}(t)}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(0,T)} &= \operatorname{diag}\left[\mathbf{n}(t)\right]^{-1} \sum_{s=0}^{T} \left(\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \operatorname{diag}\left[\boldsymbol{\theta}(s)\right] \right) &\quad (50) \\ &= \sum_{s=0}^{T} \frac{\epsilon\mathbf{n}(t)}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(s)}. &\quad (51) \end{aligned}$$

3. The response of a dependent variable integrated over time. For example, consider the population vector summed from time $t = 0$ to $t = T$. The sensitivity and elasticity of this sum are

$$\frac{d}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \sum_{t=0}^{T} \mathbf{n}(t) = \sum_{t=0}^{T} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \tag{52}$$

$$\frac{\epsilon}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(s)} \sum_{t=0}^{T} \mathbf{n}(t) = \operatorname{diag}\left[\sum_{t} \mathbf{n}(t)\right]^{-1} \sum_{t=0}^{T} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \operatorname{diag}\left[\boldsymbol{\theta}(s)\right]. \tag{53}$$

For example, Fox et al. (2001) projected the annual costs of care for Alzheimer's patients in California from 2000 to 2040. They combined projections of the population over age 65 with estimates of the prevalence of various types of Alzheimer's care and the per capita costs of such care, and transformed the annual cost figures to 1998 dollars. Although they examined the annual costs in selected years, we imagine that someone might also be interested in the accumulated expenditures, integrated over some period into the future. Denote the integrated cost from $t = 0$ to $t = T$ as

$$C(T) = \sum_{t=0}^{T} \mathbf{c}^{\mathsf{T}}(t)\mathbf{n}(t) \tag{54}$$

where $\mathbf{c}(t)$ is the (possibly time-varying) weighting vector. The sensitivity of $C(T)$

is obtained by differentiating (54):

$$\frac{dC(T)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \sum_{t=0}^{T} \mathbf{n}^{\mathsf{T}}(t) \frac{d\mathbf{c}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + \sum_{t=0}^{T} \mathbf{c}^{\mathsf{T}}(t) \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}; \tag{55}$$

it includes the possible effects of the parameters on both the cost vector and the population vector. If the weighting vector $\mathbf{c}(t)$ is fixed, the first term in (55) disappears.

4. A projection of the population of Spain

As an example, we apply the sensitivity and elasticity calculations to a projection of the population of Spain published by the Spanish Instituto Nacional de Estadística (INE). This projection is typical of cohort component projections, and Spain is an interesting case because of its recent demographic history. It has experienced rapid changes in fertility, mortality and migration during the past decades and today has one of the highest life expectancies and one of the lowest fertility levels within the European Union. During most of the 20th century, it was a country of emigration, but received large numbers of immigrants since the mid-1990s before recently again losing large population numbers to outmigration. Similar sudden changes in the vital rates may occur in the future and are difficult, perhaps impossible, to predict today. The sensitivity analyses can quantify the effects of changes in the vital rates that are not expected by INE today, but which would influence the projection results. Also, in contrast to many other statistical offices, INE has made the input data for their population projection freely available online. This allowed us to calculate the sensitivity analyses, and will also allow interested readers to replicate them.

The projection published by INE uses the cohort component method and distinguishes single-year age groups (ages 0 to 100+ years) and sex of population members. It covers a 40-year period from 2012 to 2052, with a projection interval of one year (Instituto Nacional de Estadística 2012a), based on the following assumptions:

- The fertility scenario is presented in the form of age-specific fertility rates. INE assumes that the total fertility rate will increase from 1.36 children per woman in 2011 to 1.56 in 2051, and that the mean age at childbearing will rise from 31 to 32 years within the same period. On their website, INE has published fertility vectors for $\mathbf{f}(t)$ for $t = 1, \ldots, 40$ which reflect these assumptions (Instituto Nacional de Estadística 2012b,c).
- The mortality scenario is defined in terms of the age- and sex-specific probabilities of death. It is assumed that life expectancy at birth will increase from 80 years in

2011 to 87 years in 2051 for men, and from 83 years to 91 years for women over the same time period. Corresponding to these assumptions, INE presents a series of age- and sex-specific probabilities of death, $\mathbf{q}(t)$ for $t = 1, \ldots, 40$ (Instituto Nacional de Estadística 2012b,c).

- Migration assumptions are expressed in terms of age- and sex-specific immigration numbers and emigration rates. INE assumes that the migratory balance of Spain, which was negative by 50,000 persons in 2011, will recover during the projection period. In the last ten projection years, the number of persons who move to Spain is assumed to exceed emigration numbers by around 438,000 persons. Emigration rates are held constant over the entire projection interval (Instituto Nacional de Estadística 2012b,c).[7] Because of the assumptions of INE, we incorporated emigration into the matrix $\mathbf{U}$, treating emigration and mortality as two competing risks for leaving the population. See Section 6.3 for further discussion of ways to treat migration.

In a press note on the population projections of 2012, INE emphasized two key findings. First, the population of Spain is expected to decline from 46.2 million persons in 2012 to 41.5 million by 2052. Second, the population is expected to age. INE estimates that 37 percent of the population will be aged 64 or older in 2052, raising the overall dependency ratio, defined as the quotient between the population under 16 and over 64 years of age and the population aged 16 to 64, from 0.504 (in 2012) to 0.995 (in 2052). These projection results form the basis of governmental planning (Instituto Nacional de Estadística 2012a). Analysing their sensitivity and elasticity to changes in the underlying assumptions is therefore not only relevant for the demographic research community, but also for policy makers in Spain.

5. Sensitivity and elasticity of the population projection of Spain

We investigate the sensitivity and elasticity of the projection results at the terminal time $T = 40$, considering both the population as a whole and the male and female population separately. In constructing the transition matrices $\mathbf{U}(t)$ we combined mortality and emigration as independent ways of leaving the population. Let P_i be the element in the $(i+1, i)$ entry of $\mathbf{U}$; then we write

$$P_i = (1 - q_i)\,(1 - r_i) \tag{56}$$

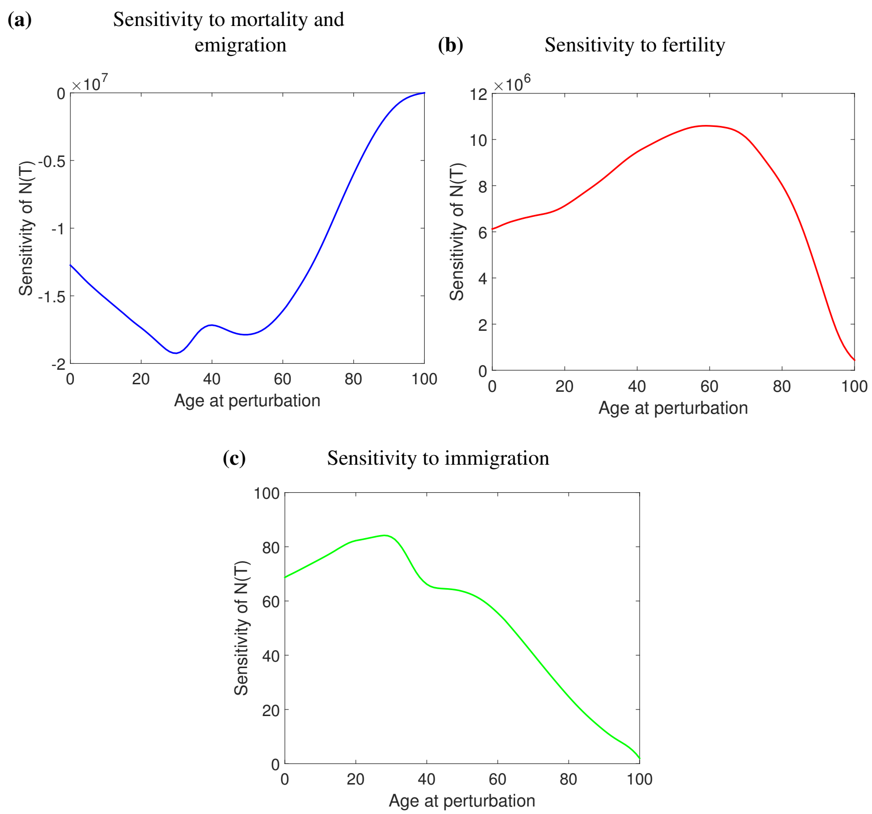
where q_i is the probability of death and r_i the probability of emigrating.

[7] This seems strange to us, but is clear in the data provided by INE.

5.1 Sensitivity of the total population size

Figure 1 shows the sensitivity of the total population size $N(T)$, at terminal time $T = 40$, to changes in the vital rates applied in every projection year, as a function of the age at which the vital rate is perturbed, as shown in equations (49) and (51).

Figure 1: Sensitivity of total population size



Notes: The sensitivity of the total population size $N(T)$, at the terminal time $T = 40$, to changes in age-specific vital rates, applied in every year from $t = 0$ to $t = T$. (a) Sensitivity to age-specific mortality or emigration, which are indistinguishable in this model. (b) Sensitivity to age-specific fertility, (c) Sensitivity to age-specific immigration. Based on Instituto Nacional de Estadística (2012c) projections for Spain from 2012 to 2052.

Figure 1 suggests that perturbations in mortality, emigration and immigration tend to have the largest effect on the final population size if they occur around age 30. Large cohorts pass through age groups 30 to 40 at the beginning of the projection period, so that any perturbations in the vital rates concern large population numbers. The effects of perturbations also accumulate during the projection period, as population members move to older age groups.

Additive perturbations in either mortality or emigration rates have a w-shaped effect on $N(T)$, with effects being largest around age 30 and to a lesser extent around age 50. Increasing rates at these ages by one unit during the projection period would reduce the final population size by between 1.8×10^7 and 2×10^7 units. Perturbations at other ages, especially above age 65, would have a smaller effect on the final population size.

Perturbations in immigration also have the strongest effect on the final population size if they occur at young adult ages. At age 30, increasing immigration numbers by one unit, i.e. by one male and one female immigrant per projection year, increases the final population size by more than 80 persons. This includes the additional immigrants themselves and their offspring. At ages above 30, the effect of perturbations in immigration numbers decreases. The sensitivities to changes in immigration are many orders of magnitude smaller than those to changes in the other vital rates. This is because immigration is measured in numbers, while mortality/emigration and fertility are per capita rates. As we will see below, elasticities help such comparisons by scaling values as proportional perturbations.

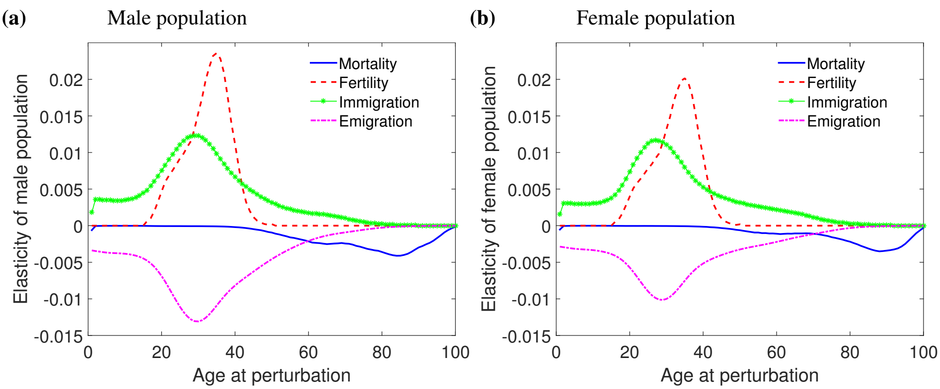
The sensitivity of total population size to perturbations in fertility rates increases with age. INE has defined fertility assumptions for women of ages 15 to 49. Among these age groups, perturbations at age 49 have the strongest effect on the total population size at time $T = 40$. An increase in fertility rates by one unit across all projection years would increase the final population size by around 10×10^6 units. Figure 1 also shows that perturbations in fertility at higher ages would have even larger effects. Unless women's fertility could be extended beyond age 50 in the future, this result is only of theoretical interest.

5.2 Elasticity of male and female population sizes

While Figure 1 compares the effects of additive perturbations across ages, comparisons between vital rates are difficult, because immigration assumptions are defined in terms of numbers and fertility and mortality/emigration assumptions as rates. In order to compare the effect of perturbations across vital rates, we calculate elasticities. Figure 2 shows the elasticity of the male and female populations at $T = 40$ to perturbations in mortality, fertility and migration as a function of the ages at which perturbations occur.

The elasticity of male and female populations to perturbations in vital rates is strongest around ages 25 to 35. The effects are stronger for the male than for the female population, because the male population reacts to perturbations of both male and female immigration numbers and emigration rates. If the female population increases, this increases the number of male offspring. The female population, by contrast, is not directly affected by perturbations in male migration. For similar reasons, the male population reacts more strongly to perturbations in fertility than the female population. The elasticity of the final male and female population sizes to perturbations in fertility is highest around age 35. The elasticity results thus confirm that projection parameters at ages 25 to 35 have to be defined with particular care if the projection outcome of interest is the final population size.

Figure 2: Elasticity of male and female population sizes



Notes: The elasticity of total male and female population size $N(T)$, at the terminal time $T = 40$, to changes in age-specific vital rates, applied in every year from $t = 0$ to $t = T$. (a) Elasticity of the total male population, (b) Elasticity of the total female population. Based on Instituto Nacional de Estadística (2012c) projections for Spain from 2012 to 2052.

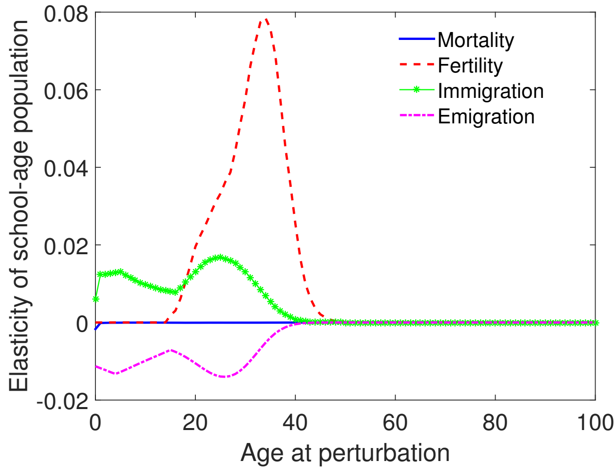
Population size is most affected by changes in mortality in old age — around 85 years for males and 90 years for females. Because mortality is low at early ages, proportional changes have little impact and the aging population increases the importance of changes affecting old individuals.

5.3 Elasticity of the school-age population (6 to 16 years)

Elasticities to changes in the vital rates can also be calculated for subgroups of the population, as in (35). As an example, we calculate the elasticity of the school-age population groups in Spain (6 to 16 years, male and female persons combined) to perturbations.

Again, we focus on the size of this population group at $T = 40$ and assume that perturbations have occurred throughout the projection period.

Figure 3: Elasticity of school-age population



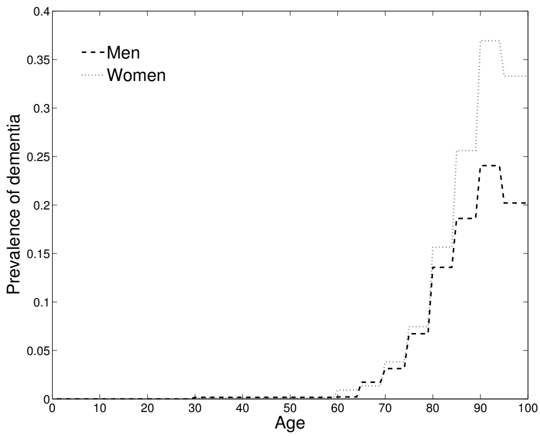
Notes: The elasticity of the school-age population size (6–16 years), at the terminal time $T = 40$, to changes in age-specific vital rates, applied in every year from $t = 0$ to $t = T$. Based on Instituto Nacional de Estadística (2012c) projections for Spain from 2012 to 2052.

Figure 3 shows that perturbations in mortality rates have almost no influence on the number of school-age children in the final projection year. Perturbations in immigration and emigration directly influence the size of the school age population if they occur at young ages (particularly ages 1 to 10 years). A one per cent increase in immigration numbers at age 5, for instance, would increase the number of school-age children in the final projection year by more than 0.01 percent. Perturbations in migration at ages 20 to 35 influence the school-age population through fertility. A change in the number of women in these age groups influences the number of newborn children in Spain who reach school age after 6 years. Fertility has by far the largest effect on the school-age population: If the fertility rate were one per cent higher than assumed by INE during the projection period at age 34 alone, the school-age population in the final projection would be 0.08 per cent larger. Fertility assumptions must therefore be of particular concern for policy makers interested in the future development of this population group.

5.4 Elasticity of population with dementia

It is often interesting to project the health status of a population. Here, we weight the population by the age-specific prevalence of dementia (Alzheimer Europe 2014) and calculate the elasticity of the number of persons with dementia at $T = 40$ to perturbations in the vital rates and in the prevalence schedule. Figure 4 shows the prevalence of dementia by age among the Spanish population in 2012. Prevalence increases strongly above age 70, and is higher for women than for men. We use these prevalences as the weight vector $\mathbf{c}$ in (35). Figure 5 shows the elasticity of the projected population with dementia in 2052 (male and female cases combined) to perturbations.[8]

Figure 4: Prevalence of dementia in Spain, 2012



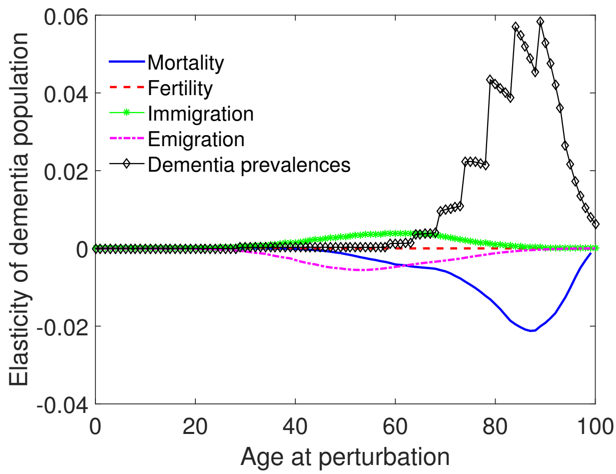
Notes: (Age- and sex-specific prevalence of dementia in Spain in the year 2012 from Alzheimer Europe (2014).

The number of persons with dementia reacts most strongly to perturbations in the prevalence schedule. A one percent increase at any age between 85 and 90 years across projection years, for instance, would increase the number of dementia cases in the last projection year by between 0.05 and 0.06 per cent. Perturbations in the vital rates have smaller effects. Changes in mortality and migration before age 30 have no effect because persons in these age groups are rarely susceptible to dementia before the end of the projection. For the same reason, changes in fertility have no effect. Above age 30, the effect

[8] We project the future number of dementia cases by holding current prevalence rates constant. For a similar approach to project the future number of dementia or Alzheimer cases, see Wancata et al. (2003) and Fox et al. (2001). For a web-based application which allows the use of time-varying incidence rates to project future Alzheimer cases, see Colantuoni et al. (2010).

of perturbations in mortality, emigration and immigration increases, reaching its highest level at ages 55 (emigration) and 60 (immigration). Perturbations of mortality show the largest impact between ages 85 and 90, when prevalence rates in dementia reach high levels. Overall, however, developments in the prevalence of dementia appear to be more decisive for the future number of dementia cases than trends in the vital rates.

Figure 5: Elasticity of the population with dementia



Notes: Elasticity of the population with dementia at the terminal time $T = 40$ to changes in age-specific vital rates and prevalences, applied in every projection year from $t = 0$ to $t = T$. Based on Instituto Nacional de Estadística (2012c) projections for Spain from 2012 to 2052 and dementia prevalence from Alzheimer Europe (2014).

5.5 Elasticity of dependency and support ratios

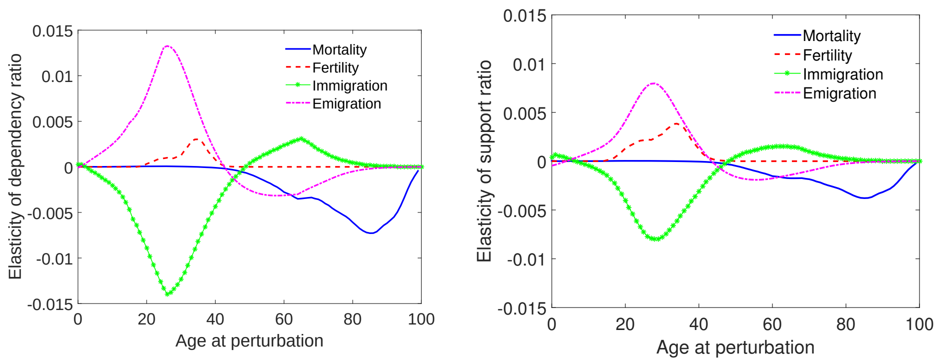
One of the findings highlighted by INE is that the dependency ratio (defining persons under age 16 and over age 64 as dependent) will double during the projection period. In 2052, the dependent population is expected to be as large as the population of working age. We show the elasticities of the dependency ratio at $T = 40$, calculated from (39) and (40), in Figure 6.

The dependency ratio reacts to perturbations in vital rates across all ages, but the size and direction of effects differ. Perturbations in immigration and emigration between ages 20 and 30, where immigration numbers and emigration rates are particularly high, have the strongest influence. Cohorts that pass through these age groups early in the projection period spend many years in the working-age population and barely contribute to to the size of the population classified as 'dependent'.

The elasticity to the mortality rate increases above age 40 and reaches a maximum around age 85. Perturbations in fertility rates have the proportionally smallest effect. This is because during the 40-year projection period, newborn cohorts contribute both to the size of age groups defined as dependent and to the working age population, and the effects cancel each other out to some extent.

Figure 6: Elasticity of dependency and support ratio

(a) Total dependency ratio **(b)** Support ratio



Notes: The elasticity of the total dependency ratio and the economic support ratio at the terminal time $T = 40$ to changes in age-specific mortality, fertility, and migration, applied in every year from $t = 0$ to $t = T$. (a) Elasticity of the total dependency ratio (dependent ages: below age 16 and above age 64), (b) Elasticity of the economic support ratio. The signs of these elasticities are reversed to make the figure comparable to the elasticities of the dependency ratio; see text for details. Based on Instituto Nacional de Estadística (2012c) projections for Spain from 2012 to 2052 and age-specific consumption and labour income data from the National Transfer Accounts Project: http://ntaccounts.org

The dependency ratio used by INE is a simplified description of economic dependency. It disregards individuals above age 65 who continue to be productive and those 16–64 who are not part of the labour force. A more nuanced perspective is provided by the economic support ratio (Prskawetz and Sambt 2014), which is the ratio of total labor income to total consumption. These ratios can be calculated from age-specific income and consumption data prepared by the National Transfer Accounts (NTA) Project.[9] We calculated support ratios for Spain, based on these NTA data for the year 2000, using per capita normalised annual consumption (public and private consumption) and labour income flows as weight vectors, as in (39).

Figure 6 shows the elasticities of the support ratio at $T = 40$ to perturbations in the vital rates, applied in every projection year.[10] The elasticities of the two indices are

[9] Data and further information are available at www.ntaccounts.org. Last accessed: 25 October 2014.

[10] The support ratio (income over consumption) has the opposite sense to the dependency ratio (dependent over

similar, but less pronounced, except for the elasticity to fertility. Increases in fertility have a larger effect on the support ratio. This reflects Spanish income and consumption patterns in 2000, in which consumption is greater than income until about age 24. Young persons therefore remain 'net consumers' for longer than assumed by the dependency ratio, and the net effect of increases in fertility would be to put a downward pressure on the support ratio.

6. Discussion

6.1 Sensitivity analysis and scenarios

Population projections are hungry for demographic data. The projection of Spain, in which 101 ages are projected over 40 years on the basis of annual rates of mortality, fertility, immigration, and emigration, contains over 16,000 pieces of information. The result of all this information is a diverse set of outcomes: population vectors, population sizes (weighted in various ways), population ratios, growth rates, etc. Changes in any of the parameters at any time will change these results. The sensitivity structure quantifies these effects.

Disciplines in which sensitivity analyses of various kinds have become common (e.g., population ecology from the 1980s onwards) experience a shift in perspective, in which the sensitivity of a dependent variable becomes as much a part of the analytical results as the dependent variable itself. From this perspective, until you have analyzed the sensitivity relationships, you have not understood the model.

Statistical offices and agencies often repeat their projections under multiple scenarios (low, medium, high, etc.). Such scenario-building is a kind of perturbation analysis, quantifying the effects of large changes imposed on many vital rates simultaneously, but the number of possible scenarios is effectively infinite. In contrast, sensitivity and elasticity analyses provide a quantitative measure of the effects of perturbations of specific rates. For example, from graphs like Figures 5 and 6, we know, without the need for any scenario modifications at all, that given changes in the vital rates would have smaller effects on the number of persons with dementia than would changes in the prevalence rates. We know that changes in fertility will have different effects on the economic support ratio than on the total dependency ratio. Such conclusions may help decide what kind of scenario modifications are most worth examining.

productive). To facilitate comparisons between the two measures, we have reversed the signs of the elasticity results for the economic support ratio.

6.2 Sensitivity and uncertainty

Because population projections are used for social, economic, and ecological planning, demographers have invested considerable attention to measuring their uncertainty. A large body of literature has focused on probabilistic population projections based on past projection errors, expert opinion or stochastic models (Keilman, Pham, and Hetland 2002).

Sensitivity analysis does not, by itself, provide information on the uncertainty of a projection [it is a prospective, not a retrospective, perturbation analysis, in the terminology of Caswell (2000)]. Knowing that an outcome is more or less sensitive to some parameter does not tell whether the outcome is more or less certain. That also depends on the precision with which the parameter is estimated.

When this precision is known, sensitivity analysis provides a powerful way to translate it into the uncertainty in projection outcomes. Suppose that $\boldsymbol{\xi}$ is a projection result (vector- or scalar-valued) that depends on a set of parameters $\boldsymbol{\theta}$. The uncertainty in the estimate of $\boldsymbol{\xi}$ is measured by the covariance matrix

$$\mathbf{C}(\boldsymbol{\xi}) = \left(\; \mathrm{Cov}(\xi_i, \xi_j) \; \right). \tag{57}$$

If ξ is a scalar, this is simply the variance $V(\xi)$.

The uncertainty in the parameter estimates is quantified by the covariance matrix $\mathbf{C}(\boldsymbol{\theta})$, which might be obtained, e.g., from the Fisher information matrix provided by maximum likelihood estimation of $\boldsymbol{\theta}$.

Then, to first order, the uncertainty in $\boldsymbol{\theta}$ translates into uncertainty in $\boldsymbol{\xi}$ by

$$\mathbf{C}(\boldsymbol{\xi}) = \frac{d\boldsymbol{\xi}}{d\boldsymbol{\theta}^{\mathsf{T}}} \, \mathbf{C}(\boldsymbol{\theta}) \, \left(\frac{d\boldsymbol{\xi}}{d\boldsymbol{\theta}^{\mathsf{T}}} \right)^{\mathsf{T}}. \tag{58}$$

If ξ is a scalar, this reduces to

$$V(\xi) = \frac{d\xi}{d\boldsymbol{\theta}^{\mathsf{T}}} \, \mathbf{C}(\boldsymbol{\theta}) \, \left(\frac{d\xi}{d\boldsymbol{\theta}^{\mathsf{T}}} \right)^{\mathsf{T}} \tag{59}$$

and if θ is also a scalar, then

$$V(\xi) = \left(\frac{d\xi}{d\theta} \right)^2 V(\theta). \tag{60}$$

These calculations formalize the intuitive notion that uncertainty in a parameter to which an outcome is very sensitive translates into a high degree of uncertainty in that outcome, and the sensitivities $d\boldsymbol{\xi}/d\boldsymbol{\theta}^{\mathsf{T}}$ are essential to the translation.

6.3 Immigration and emigration

Migration is challenging to model. Births, deaths, and emigration are events that happen to individuals in the population under study. They can be described by rates, estimated from the number of events and the number of individuals at risk. Those rates can be transformed to probabilities and then applied to the appropriate components of cohorts to project the population forward.

Immigration, however, is not an event to which individuals in the population are at risk, and hence it cannot be described as a rate. Thus, in equations (4), (7), and (8), immigration appears as a vector $\mathbf{b}(t)$, with units of numbers of individuals, rather than as one of the per capita rates in $\mathbf{U}$ and $\mathbf{F}$.

Immigration is handled in various ways by the agencies engaged in projections. The projection of Spain in Section 4 takes the sensible approach of separating emigration and immigration, including emigration along with mortality in the matrix $\mathbf{U}$, and placing immigration in $\mathbf{b}(t)$. The projections prepared by Eurostat (Lanzieri 2009) make this approach slightly more subtle, noting that individuals who immigrate during $(t, t+1)$ spend some fraction of the interval in the population, and hence subject to the mortality and fertility rates in action during that time (G. Lanzieri, personal communication). This means that a basic projection equation becomes

$$\mathbf{n}(t+1) = \mathbf{A}(t)\mathbf{n}(t) + \mathbf{B}(t)\mathbf{b}(t) \tag{61}$$

where $\mathbf{B}(t)$ is a matrix that includes mortality and fertility of immigrants during the fraction of the interval during which they are assumed to be present (usually 0.5 years). Our approach is easily extended to the projection (61), for example, simply by replacing the term $d\mathbf{b}(t)/d\boldsymbol{\theta}^{\mathsf{T}}(s)$ in equation (13) with

$$\mathbf{B}(t)\frac{d\mathbf{b}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + \left(\mathbf{b}^{\mathsf{T}}(t) \otimes \mathbf{I}_\omega\right) \frac{d\mathrm{vec}\,\mathbf{B}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}.$$

A different approach defines the additive vector $\mathbf{b}$ as *net* migration (immigration $-$ emigration), thus treating both immigration and emigration as additive. This has unfortunate theoretical properties; it asserts that the number of individuals leaving the population is independent of the population at risk of leaving. In principle, in the long run this could draw a population down to impossible negative values. For the short time horizons in practical population projections, this is unlikely to be a problem.

Yet another option describes both immigration and emigration as rates applied to the population at risk. This conceptualizes immigration as a flow of individuals “sucked” into the population by the residents. It also has bad long-run theoretical properties: the number of immigrants goes to zero as population decreases, and increases without bounds as the population grows.

These models focus on a single population. The analysis of migration can also be embedded in a multiregional model (Rogers 1975), in which the immigrants to one population come from the emigrants leaving another population. Such multiregional models are a special case of multistate models (Rogers 1985) and the beginnings of the theory for sensitivity analysis of multistate models exists (Caswell 2012; Caswell and Salguero-Gomez 2013). We will explore the sensitivity analysis of multistate projections in a subsequent paper.

All of these approaches produce linear projections that are simple modifications of the basic models (4)–(8). There is also a tradition of theories that model migration as a function of population size, distance, and other properties of origin and destination. These models are descendants of the Zipf (1946) gravity model. Courgeau (1995) discusses how these models arise from theories of choice. They lead to expressions in which the log of the number of migrants from location i to location j is a linear function of the logs of population size, distance, and other variables. Cohen et al. (2008) and Kim and Cohen (2010) have applied the method in a detailed analysis of data on international migration and were able to capture significant amounts of the variance in migration flows.

Because migration in gravity models is a power function of population size, the resulting projections are nonlinear. We do not address the analysis here, but note that sensitivity analysis can also be applied to nonlinear models (Caswell 2007, 2008), a development noted by Cohen et al. (2008) as an open research question.

6.4 Data requirements and applications

Goldstein and Stecklov (2002) have lamented the lack of clarity and transparency in reports of population projections. The trajectories of mortality, fertility, and immigration on which the projections depend are seldom reported, and "even when extensive documentation is provided, it is difficult to replicate the calculations without access to proprietary computer software used by the team that prepared the projection" (Goldstein and Stecklov 2002, p. 121). We urge agencies to consider reporting the basis of their projections in the form of projection matrices. The entries of $\mathbf{U}$, $\mathbf{F}$, and $\mathbf{b}$ may require considerable effort to obtain, and sophisticated methods may be needed to estimate them from data on populations, births, deaths, etc. But once the estimation process is completed, the projection matrix formulation provides a readily computable, non-proprietary method of studying the results and exploring scenarios. The mathematical relationships extracted from those matrices will be valid regardless of how the matrices themselves are obtained. Sensitivity analysis is just one of their possible uses.

Sensitivity analyses using matrix calculus techniques require only the basic ingredients of any cohort component projection: the initial age- and sex-specific population vectors and the fertility, mortality, and migration parameters for each projection year. The sensitivity and elasticity analyses can be extended to multistate population projections; these developments are left for future research. In the meantime, the analyses presented here will benefit demographers and government officials producing projections, because they will improve our understanding of the underlying mechanisms leading to uncertainties and allow for precise quantifications of the impact of changes in vital rates or policies on any projection output.

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Appendix A. Derivations

In this section, we present the derivations of the sensitivity results in Section 3. The derivations use matrix calculus (Magnus and Neudecker 1985, 1988). For a general presentation of the approach, and many additional demographic applications of this approach, see Caswell (2008, 2009). For an introductory presentation of the mathematics of matrix calculus, see Abadir and Magnus (2005). Early, but not well known, demographic papers using matrix calculus for sensitivity analyses include Willekens (1977), and Ekamper and Keilman (1993). Some related ideas, couched in terms of functional analysis, can be found in Arthur (1984).

To obtain derivatives, we begin by calculating differentials. The differential of a matrix $\mathbf{X}$ is the matrix of differentials of the elements of $\mathbf{X}$:

$$d\mathbf{X} = \left(\; dx_{ij} \; \right). \tag{A-1}$$

These differentials follow rules familiar from scalar calculus; in particular we will make use of the product rule: for any two matrices $\mathbf{X}$ and $\mathbf{Y}$,

$$d(\mathbf{X}\mathbf{Y}) = (d\mathbf{X})\mathbf{Y} + \mathbf{X}(d\mathbf{Y}). \tag{A-2}$$

We make frequent use of the vec operator, which creates a vector from a matrix by stacking columns on top of each other; e.g.,

$$\mathrm{vec} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c & b & d \end{pmatrix}^{\mathsf{T}}. \tag{A-3}$$

An important result, due to Roth (1934), says that, for any matrices $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ of the proper sizes to multiply,

$$\mathrm{vec}\, \mathbf{X}\mathbf{Y}\mathbf{Z} = (\mathbf{Z}^{\mathsf{T}} \otimes \mathbf{X})\, \mathrm{vec}\, \mathbf{Y}. \tag{A-4}$$

The vec of a matrix differential satisfies

$$\mathrm{vec}\; (d\mathbf{X}) = d\mathrm{vec}\, \mathbf{X}. \tag{A-5}$$

A.1 Derivatives of $\mathbf{n}(t)$

One-sex projections. We begin with the single sex projection of equation (4). Take the differential of both sides to obtain

$$d\mathbf{n}(t+1) = \mathbf{A}(t)d\mathbf{n}(t) + \left[d\mathbf{A}(t)\right]\mathbf{n}(t) + d\mathbf{b}(t). \tag{A-6}$$

Now apply the vec operator to both sides. If we write $\left[d\mathbf{A}(t)\right]\mathbf{n}(t) = \mathbf{I}_\omega \left[d\mathbf{A}(t)\right]\mathbf{n}(t)$, then Roth's theorem implies that

$$d\mathbf{n}(t+1) = \mathbf{A}(t)d\mathbf{n}(t) + \left(\mathbf{n}^{\mathsf{T}}(t) \otimes \mathbf{I}_\omega\right) d\mathrm{vec}\, \mathbf{A}(t) + d\mathbf{b}(t). \tag{A-7}$$

Notice that $\mathbf{A}(t)$ and $\mathbf{b}(t)$, at time t, are functions of the parameter vector $\boldsymbol{\theta}(s)$, at time s. By the chain rule for matrix calculus, the derivative with respect to the $\boldsymbol{\theta}(s)$ is then

$$\frac{d\mathbf{n}(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \mathbf{A}(t)\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + \left(\mathbf{n}^{\mathsf{T}}(t)\otimes\mathbf{I}_{\omega}\right)\frac{d\mathrm{vec}\,\mathbf{A}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} + \frac{d\mathbf{b}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)}. \tag{A-8}$$

This is a dynamic system in the derivative matrix $d\mathbf{n}(t)/d\boldsymbol{\theta}^{\mathsf{T}}(s)$. If the parameter vector affects the vital rates but not the starting population for the projection, then (A-8) is iterated from the initial condition

$$\frac{d\mathbf{n}(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \mathbf{0}_{\omega\times p}. \tag{A-9}$$

The sensitivity of the projection to the initial population is obtained by setting $\boldsymbol{\theta} = \mathbf{n}_0$. The last two terms in (A-8) are then zero, and the remaining term is iterated from the initial condition

$$\frac{d\mathbf{n}(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \mathbf{I}_{\omega}. \tag{A-10}$$

Two-sex projections. We apply the same approach to the two-sex projection in equations (7) and (8). For notational convenience, we temporarily suppress the time-dependence of the matrices $\mathbf{U}(t)$, $\mathbf{F}(t)$, and $\mathbf{b}(t)$. Differentiating both sides of (7) and (8) gives

$$\begin{aligned} d\mathbf{n}_f(t+1) &= (d\mathbf{U}_f)\,\mathbf{n}_f(t) + \mathbf{U}_f d\mathbf{n}_f(t) + \phi\,(d\mathbf{F})\,\mathbf{n}_f \\ &\quad +\phi\mathbf{F}d\mathbf{n}_f(t) + d\mathbf{b}_f & \text{(A-11)} \\ d\mathbf{n}_m(t+1) &= \mathbf{U}_m d\mathbf{n}_m(t) + (1-\phi)\mathbf{F}d\mathbf{n}_f(t) + (d\mathbf{U}_m)\,\mathbf{n}_m(t) \\ &\quad +(1-\phi)\,(d\mathbf{F})\,\mathbf{n}_f(t) + d\mathbf{b}_m. & \text{(A-12)} \end{aligned}$$

Applying the vec operator gives

$$\begin{aligned} d\mathbf{n}_f(t+1) &= (\mathbf{U}_f + \phi\mathbf{F})\,d\mathbf{n}_f(t) + \left(\mathbf{n}_f^{\mathsf{T}}(t)\otimes\mathbf{I}_{\omega}\right)d\mathrm{vec}\,\mathbf{U}_f \\ &\quad +\phi\left(\mathbf{n}_f^{\mathsf{T}}(t)\otimes\mathbf{I}_{\omega}\right)d\mathrm{vec}\,\mathbf{F} + d\mathbf{b}_f & \text{(A-13)} \end{aligned}$$

$$\begin{aligned} d\mathbf{n}_m(t+1) &= \mathbf{U}_m d\mathbf{n}_m(t) + (1-\phi)\mathbf{F}d\mathbf{n}_f(t) + \left(\mathbf{n}_m^{\mathsf{T}}(t)\otimes\mathbf{I}_{\omega}\right)d\mathrm{vec}\,\mathbf{U}_m \\ &\quad +(1-\phi)\left(\mathbf{n}_f^{\mathsf{T}}(t)\otimes\mathbf{I}_{\omega}\right)d\mathrm{vec}\,\mathbf{F} + d\mathbf{b}_m. & \text{(A-14)} \end{aligned}$$

Notice that the male population is sensitive to changes in the parameters of the female population, because fertility of females produces new males. The second and fourth terms in (A-14) provide the required links between the female and male population.

We introduce the parameter vector $\boldsymbol{\theta}(s)$ and use the chain rule to write the differentials in (A-13) and (A-14) as derivatives with respect to $\boldsymbol{\theta}(s)$, and thus obtain the sensitivity expressions (17) and (18).

A.2 Derivatives of projection matrices

We turn now to the derivatives with respect to mortality, fertility, and immigration of the projection matrices $\mathbf{U}$ and $\mathbf{F}$, and the immigration vector $\mathbf{b}$, given in Section 3.4.

Mortality. Write the matrix $\mathbf{U}$ as

$$\mathbf{U} = \mathbf{Z} \circ \left(\mathbf{1}_\omega \mathbf{p}^\mathsf{T}\right), \tag{A-15}$$

where $\mathbf{Z}$ contains ones on the subdiagonal and zeros elsewhere, and $\mathbf{p} = \exp(-\boldsymbol{\mu})$ is a vector of survival probabilities. Differentiating gives

$$d\mathbf{U} = \mathbf{Z} \circ \left(\mathbf{1}_\omega d\mathbf{p}^\mathsf{T}\right). \tag{A-16}$$

Apply the vec operator to obtain

$$d\text{vec}\,\mathbf{U} = \text{diag}\left(\text{vec}\,\mathbf{Z}\right)\text{vec}\,\left(\mathbf{1}_\omega d\mathbf{p}^\mathsf{T}\right) \tag{A-17}$$

$$= \text{diag}\left(\text{vec}\,\mathbf{Z}\right)\left(\mathbf{I}_\omega \otimes \mathbf{1}_\omega\right) d\mathbf{p} \tag{A-18}$$

where the differential of $\mathbf{p}$ is

$$d\mathbf{p} = -\text{diag}\left(\mathbf{p}\right) d\boldsymbol{\mu}. \tag{A-19}$$

Substituting (A-19) into (A-18) gives the result (24).

Fertility. The matrix $\mathbf{F}$, with the fertilities $\mathbf{f}$ in the first row and zeros elsewhere, can be written

$$\mathbf{F} = \mathbf{e}_1 \mathbf{f}^\mathsf{T} \tag{A-20}$$

where $\mathbf{e}_1$ is the first unit vector of dimension ω. Differentiating gives

$$d\mathbf{F} = \mathbf{e}_1 d\mathbf{f}^\mathsf{T}. \tag{A-21}$$

Applying the vec operator gives

$$d\text{vec}\,\mathbf{F} = \left(\mathbf{I}_\omega \otimes \mathbf{e}_1\right) d\mathbf{f} \tag{A-22}$$

which appears in (25).

Immigration. The derivative of the immigration vector to itself is the identity matrix, by definition.

A.3 Derivatives of dependent variables

The derivative of a weighted population size follows from differentiating $N = \mathbf{c}^{\mathsf{T}}\mathbf{n}(t)$,

$$dN = \mathbf{c}^{\mathsf{T}} d\mathbf{n}(t) \tag{A-23}$$

from which it follows that

$$\frac{dN}{d\mathbf{n}^{\mathsf{T}}(t)} = \mathbf{c}^{\mathsf{T}}. \tag{A-24}$$

The chain rule,

$$\frac{dN}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} = \frac{dN}{d\mathbf{n}^{\mathsf{T}}(t)} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(s)} \tag{A-25}$$

gives equation (35).

To obtain the sensitivity of a ratio $R(t)$, differentiate (38),

$$dR(t) = \frac{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\mathbf{a}^{\mathsf{T}}d\mathbf{n}(t) - \mathbf{a}^{\mathsf{T}}\mathbf{n}(t)\mathbf{c}^{\mathsf{T}}d\mathbf{n}(t)}{\left[\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\right]^2}. \tag{A-26}$$

Factoring out $d\mathbf{n}(t)$ and applying the chain rule gives equation (39).

The sensitivity of the growth rate $\lambda(t)$ is obtained by differentiating equation (44),

$$d\lambda(t) = \frac{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\mathbf{c}^{\mathsf{T}}d\mathbf{n}(t+k) - \mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)\mathbf{c}^{\mathsf{T}}d\mathbf{n}(t)}{\left[\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\right]^2} \tag{A-27}$$

$$= \frac{\mathbf{c}^{\mathsf{T}}}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} d\mathbf{n}(t+k) - \frac{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)\mathbf{c}^{\mathsf{T}}}{\left[\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\right]^2} d\mathbf{n}(t) \tag{A-28}$$

$$= \frac{\mathbf{c}^{\mathsf{T}}}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} d\mathbf{n}(t+k) - \frac{\lambda(t)\mathbf{c}^{\mathsf{T}}}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} d\mathbf{n}(t) \tag{A-29}$$

from which (45) follows by the chain rule.

Appendix B. Notes on the treatment of immigration in population projections

From an informal survey of Statistical Offices in European countries and the United Nations, we found migration to be described in terms of either net migration numbers or in terms of immigration numbers and emigration rates.

1. Net migration numbers: the Statistical Offices of Germany, France, Ireland, Estonia, Sweden, Switzerland, Iceland, the United Kingdom, and the United Nations use net migration numbers. These are partly defined on the basis of past developments of net migration trends; some Statistical Offices also identify potential future trends in immigration and emigration, and then deduct emigration numbers from immigration numbers for each projection year. In our notation, the resulting net migration numbers would appear in the vector $\mathbf{b}(t)$.
2. Immigration numbers and emigration rates: The Statistical Offices of Austria and Belgium have chosen the same approach as the Statistical Office of Spain in defining emigration as a rate and including it with mortality in the matrix $\mathbf{U}(t)$. Immigration is defined in terms of numbers and included in $\mathbf{b}(t)$.

The Statistical Offices of Germany, France, Sweden, and the United Kingdom state that half of all immigrants and emigrants are assumed to move at the beginning of each projection year, and half at the end of each projection year; or that all migrants spend half of the immigration/emigration year in the population. Migrants are subject to the mortality and fertility schedules during the fraction of the projection year that they spend in the country. The United Nations projections follow a similar approach.

In addition to their treatment of immigration, statistical offices differ in terms of the geographical level used for the projections and in terms of the number of criteria by which the projections are planned and calculated. Some offices project the population on a country-level. Others calculate projections for subnational regions or municipalities and combine them to project the entire country. The projections of Switzerland and Sweden are multistate projections, with separate projections calculated for citizens and foreign residents. Table B-1 provides links to websites of the statistical offices discussed here.

Table B-1: Links to websites of statistical offices.

- Austria: http://www.statistik.at/web_de/statistiken/bevoelkerung/demographische_prognosen/bevoelkerungsprognosen/index.html,
- Belgium: http://statbel.fgov.be/fr/binaries/FORPOP1360_10697_F_tcm326-244744.pdf,
- Estonia: http://www.stat.ee/76320,
- France: http://www.insee.fr/fr/publications-et-services/docs_doc_travail/docf1008.pdf,
- Germany: https://www.destatis.de/DE/ZahlenFakten/GesellschaftStaat/Bevoelkerung/Bevoelkerungsvorausberechnung/VorausberechnungsModell.pdf?__blob=publicationFile,
- Iceland: http://www.statice.is/lisalib/getfile.aspx?ItemID=15409,
- Ireland: http://www.cso.ie/en/media/csoie/releasespublications/documents/population/2013/poplabfor2016_2046.pdf,
- Switzerland: http://www.bfs.admin.ch/bfs/portal/de/index/news/publikationen.html?publicationID=3989,
- Sweden: http://www.scb.se/statistik/_publikationer/BE0401_2012I60_BR_BE51BR1202ENG.pdf,
- United Kingdom: http://www.ons.gov.uk/ons/rel/npp/national-population-projections/2012-based-reference-volume–series-pp2/index.html,
- United Nations: http://www.un.org/en/development/desa/population/publications/manual/projection/sex-age.shtml,