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Gamma-Gompertz life expectancy at birth

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Abstract

BACKGROUND
The gamma-Gompertz multiplicative frailty model is the most common parametric model applied to human mortality data at adult and old ages. The resulting life expectancy has been calculated so far only numerically.

OBJECTIVE
Properties of the gamma-Gompertz distribution have not been thoroughly studied. The focus of the paper is to shed light onto its first moment or, demographically speaking, characterize life expectancy resulting from a gamma-Gompertz force of mortality. The paper provides an exact formula for gamma-Gompertz life expectancy at birth and a simpler high-accuracy approximation that can be used in practice for computational convenience. In addition, the article compares actual (life-table) to model-based (gamma-Gompertz) life expectancy to assess on aggregate how many years of life expectancy are not captured (or overestimated) by the gamma-Gompertz mortality mechanism.

COMMENTS
A closed-form expression for gamma-Gomeprtz life expectancy at birth contains a special (the hypergeometric) function. It aids assessing the impact of gamma-Gompertz parameters on life expectancy values. The paper shows that a high-accuracy approximation can be constructed by assuming an integer value for the shape parameter of the gamma distribution. A historical comparison between model-based and actual life expectancy for Swedish females reveals a gap that is decreasing to around 2 years from 1950 onwards. Looking at remaining life expectancies at ages 30 and 50, we see this gap almost disappearing.

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1. Relationship

Suppose in a population individuals die according to a force of mortality

$$\mu(x \mid Z) = Z\mu(x), \tag{1}$$

where Z is a random variable, called frailty (Vaupel, Manton, and Stallard 1979), which accounts for unobserved heterogeneity across individuals, and $\mu(x)$ is the baseline force of mortality. Model (1) is called a multiplicative (frailty) model.

Assume $\mu(x)$ follows the Gompertz law

$$\mu(x) = ae^{bx}\,, \qquad a, b > 0$$

and frailty is gamma-distributed, i.e. $Z \sim \Gamma(k, \lambda)$ has a probability density function

$$\pi(z) = \frac{\lambda^k}{\Gamma(k)}\, z^{k-1}\, e^{-\lambda x}\,, \qquad k, \lambda > 0\,.$$

Then life expectancy at birth e_0 can be expressed as

$$e_0 = \frac{1}{bk}\, {}_2F_1\left(k, 1; k+1; 1 - \frac{a}{b\lambda}\right)\,, \tag{2}$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is the Gaussian hypergeometric function

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{j=0}^{+\infty} \frac{\alpha(\alpha+1)\ldots(\alpha-j+1)\,\beta(\beta+1)\ldots(\beta-j+1)}{\gamma(\gamma+1)\ldots(\gamma-j+1)\,j!}\, z^j \tag{3}$$

defined for $\gamma > \beta > 0$ (see, for example, Abramowitz and Stegun 1965, 15.1.1, p.556). If the shape parameter k of the gamma distribution is an integer, then

$$e_0 = \frac{1}{b}\left[\left(1 - \frac{a}{b\lambda}\right)^{-k} \ln\frac{b\lambda}{a} - \sum_{j=1}^{k-1} \frac{1}{j}\left(1 - \frac{a}{b\lambda}\right)^{j-k}\right] \tag{4}$$

and, moreover, (4) serves as a high-accuracy approximation of (2) even when k is not integer. As a result, gamma-Gompertz life expectancy can be calculated by using (4).

2. Proof of the Relationship

Life expectancy is the integrated survivorship of the population across all ages

$$e_0 = \int_0^\infty S(x)dx$$

where

$$S(x) = \int_0^\infty \exp\left\{ -\int_0^x \mu(t \,|\, z)dt \right\} \pi(z)dz$$

In a gamma-Gompertz multiplicative model $S(x) = \left(1 + \frac{a}{b\lambda}\left(e^{bx} - 1\right)\right)^{-k}$ and thus

(5) $$e_0 = \int_0^\infty \left(1 + \frac{a}{b\lambda}\left(e^{bx} - 1\right)\right)^{-k} dx.$$

A $t = 1 - e^{-bx}$ substitution will result in

(6) $$e_0 = \frac{1}{b}\int_0^1 (1-t)^{k-1}\left(1 - \left(1 - \frac{a}{b\lambda}\right)t\right)^{-k} dt$$

Taking into account

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1}\,(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt$$

(see Abramowitz and Stegun 1965, 15.3.1, p.558), (6) reduces to (2). Relationship (4) is obtained by integrating k times the right-hand side of (5) by parts.

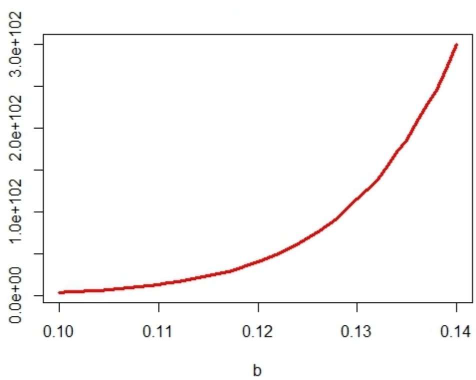
Q.E.D.

3. History and Related Results

The gamma-Gompertz multiplicative frailty model (1) has been introduced in demography by Vaupel, Manton, and Stallard (1979). While capturing the observed bending of human mortality rates at older ages (see Beard 1959), it also takes into account unobserved heterogeneity. Relationship (2) describes, on the one hand, the first moment of the mixture gamma-Gompertz distribution and, from a demographic point of view, the expected lifetime duration under the gamma-Gompertz assumption.

The fact that gamma-Gompertz life expectancy is proportional to a hypergeometric function with a z-argument close to 1 (for human populations $a \propto 10^{-6}$, $b \approx 0.14$, and $k = \lambda > 1$) sheds light on the dynamics of e_0 with respect to model parameters. As $\gamma - \alpha - \beta = 0$ for the hypergeometric function ${}_2F_1$ in (2), life expectancy is an increasing function of z for $z \to 1-$ and $\lim\limits_{z \to 1} {}_2F_1(\alpha, \beta; \gamma; z) = +\infty$ (see Abramowitz and Stegun 1965, p.556, 15.1.1(c)). This implies that e_0 increases when a declines keeping all other parameters fixed, which is intuitively justified as a denotes the starting level of mortality. A little counterintutive is the finding that life expectancy increases as the rate of aging $b = d \ln \mu(x)/dx$ increases (see Figure 1). The gamma parameters k and λ, often assumed to be equal to one another, so that $\mu(x)$ denotes the force of mortality of the "standard" individual (with frailty $Z = 1$), have one and the same impact on life expectancy – the higher k, λ, the higher e_0.

Figure 1: Life expectancy as a function of b.



Notes: Life expectancy at birth as a function of b for fixed $a = 5 \times 10^{-7}$, $k = \lambda = 7$.

Note that when baseline mortality $\mu(x)$ in (1) is Gompertz-Makeham, i.e.

$$\mu(x) = ae^{bx} + c\,,$$

and, consequently,

$$\mu(x \mid Z) = Z(ae^{bx} + c)\,,$$

the corresponding life-expectancy integral

$$e_0 = \int_0^\infty \left(1 - \frac{a}{b\lambda} + \frac{c}{\lambda}x + \frac{a}{b\lambda}e^{bx}\right)^{-k} dx$$

cannot be solved analytically, even for integer values of k.

As already pointed out, for human populations we have $1 - a/b\lambda \approx 1$, which leads to further simplification of (4):

$$e_0 \approx \frac{1}{b}\left[\ln\frac{b\lambda}{a} - \sum_{j=1}^{k-1}\frac{1}{j}\right] \tag{7}$$

Note that when $k = \lambda$ (often assumed, see Vaupel, Manton, and Stallard (1979)), the right-hand side of (7) contains the difference between the partial sum of the harmonic series and the natural logarithm:

$$e_0 \approx \frac{1}{b}\left[\ln\frac{b}{a} + \frac{1}{k} - \left(\sum_{j=1}^{k}\frac{1}{j} - \ln k\right)\right]\,. \tag{8}$$

The limit of the latter when $k \to \infty$ is the Euler-Mascheroni constant $\gamma^* \approx 0.577$. Note that $k \to \infty$ corresponds to the case when the Gompertz model for a gamma-heterogeneous population tends to the Gompertz model for a homogeneous population. As a result, life expectancy at birth for a homogeneous population experiencing a Gompertz force of mortality could be approximated by

$$e_0 \approx \frac{1}{b}\left[\ln\frac{b}{a} - \gamma^*\right]\,. \tag{9}$$

An expression for remaining life expectancy e_x at age x in terms of a hypergeometric function can also be derived. Consider the indefinite integral in the right-hand side of (5):

$$I := \int \left(1 + \frac{a}{b\lambda}\left(e^{bx} - 1\right)\right)^{-k} dx = \int \left(\frac{b\lambda}{a}e^{-bx}\right)^{k} \left(1 - \left(1 - \frac{b\lambda}{a}\right)e^{-bx}\right)^{-k} dx\,.$$

A $y = e^{-bx}$ substitution will result in

$$I = -\frac{1}{b}\left(\frac{b\lambda}{a}\right)^{k} \int y^{k-1}\left(1 - \left(1 - \frac{b\lambda}{a}\right)y\right)^{-k} dy\,.$$

Taking into account (see Lebedev 1965, p.258)

$$\left(1 - \left(1 - \frac{b\lambda}{a}\right)y\right)^{-k} = {}_2F_1\left(k, C; C; \left(1 - \frac{b\lambda}{a}\right)y\right) \qquad \forall C \equiv const$$

and ${}_2F_1(\alpha, \beta; \gamma; z) = {}_2F_1(\beta, \alpha; \gamma; z)$, we have

$$\left(1 - \left(1 - \frac{b\lambda}{a}\right)y\right)^{-k} = {}_2F_1\left(k, k; k; \left(1 - \frac{b\lambda}{a}\right)y\right)\,.$$

Using in addition (see MathWorld 2012, http://functions.wolfram.com/07.23.21.0006.01)

$$\int z^{\gamma-1}\,{}_2F_1(\alpha, \beta; \gamma; z)dz = \frac{z^{\gamma}}{\gamma}\,{}_2F_1(\alpha, \beta; \gamma+1; z)$$

and switching back to the original variable x, we reduce (6) to

$$I = -\frac{1}{bk}\left(\frac{b\lambda}{a}e^{-bx}\right)^{k}\,{}_2F_1\left(k, k; k+1; \left(1 - \frac{b\lambda}{a}\right)e^{-bx}\right)\,.$$

Taking into account

$$\lim_{x\to\infty}\left\{-\frac{1}{bk}\left(\frac{b\lambda}{a}e^{-bx}\right)^{k}\,{}_2F_1\left(k, k; k+1; \left(1 - \frac{b\lambda}{a}\right)e^{-bx}\right)\right\} = 0\,,$$

we finally get

$$e_x = \frac{1}{bk}\left(\frac{b\lambda}{a}e^{-bx}\right)^{k}\,{}_2F_1\left(k, k; k+1; \left(1 - \frac{b\lambda}{a}\right)e^{-bx}\right)\,. \tag{10}$$

4. Applications

Relationship (2) and its approximation (4) can be used to measure the difference between actual (calculated by lifetable methods) and model-predicted (based on the estimation of gamma-Gompertz parameters) life expectancy at birth. This difference quantifies the cumulative excess infant and adult mortality. Figure 2 illustrates this gap for Swedish females from 1891 to 2010. As infant mortality improves over time, the difference decreases from 1950 onwards to an almost constant value of about 2 years.

If we use expression (10) or its approximation (for integer k) analogous to (4), we can see that the gap between actual and fitted gamma-Gompertz remaining life expectancy decreases over age x (see Figure 3). This illustrates on aggregate the phenomenon that most deaths which are not captured by the gamma-Gompertz model, occur from infant to young adult ages.

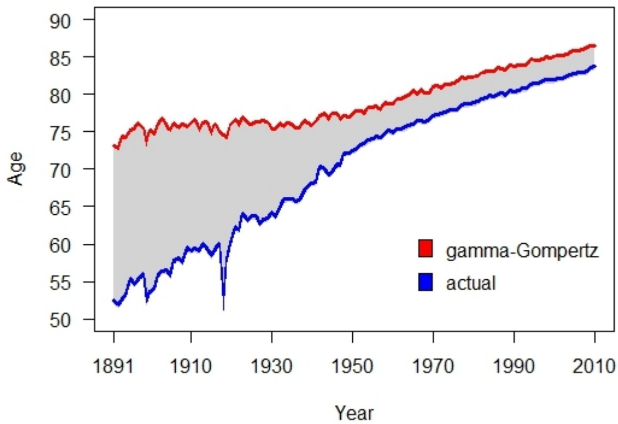
Empirically, it does not make a significant difference whether life expectancy is calculated in terms of the hypergeometric function (2) or by approximation (4). I use the data for Swedish females (HMD 2012) to estimate parameters a, b, and k (assuming $k = \lambda$) by miximizing a Poisson likelihood of the respective death counts. I start at age 70, assuming the baseline force of mortality onwards to be purely Gompertz, and calculate the initial mortality level by multiplying the estimated $\hat{a}$ by $\exp\{(\text{initial age} - 70)\hat{b}\}$, where $\hat{b}$ is the maximum-likelihood estimate of b. Table 1 shows observed and fitted gamma-Gompertz life expectancies at birth and at age 30 for Swedish females in several specified years, illustrating how close these values are, regardless of the proximity of $\hat{k}$ to its closest integer $[\hat{k}]$. This implies that one can use (4) instead of (2) without losing much precision.

Approximations (7) and (8) are not very accurate as small deviations a/bk ($k = \lambda$) from 1 can lead to substantial deviations from (4) and, thus, from (2). They can be used, though, to assess the impact of model parameters on the values of life expectancy at birth.

5. Conclusion

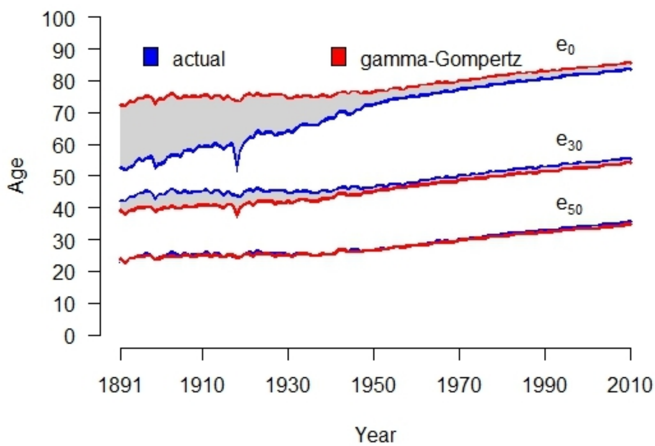
Life expectancy in a gamma-Gompertz multiplicative model can be expressed analytically in terms of a special function (the hypergeometric series), which provides insight on life expectancy dynamics with respect to model parameters. In practice, one can use high-accuracy approximation (4) instead of (2) to calculate model-based e_0 for fitted parameter values. The difference between the latter and actual (life-table) life expectancy at birth or any age x could tell how many years of actual life expectancy are potentially due to causes not captured by the gamma-Gompertz frailty model.

Figure 2: Actual vs gamma-Gompertz life expectancy at birth.



Notes: Actual vs gamma-Gompertz life expectancy (Data source: HMD (2012), Sweden, females; own estimation).

Figure 3: Actual vs gamma-Gompertz life expectancy.



Notes: Actual vs gamma-Gompertz remaining life expectancy at ages 0, 30, and 50 (Data source: HMD (2012), Sweden, females; own estimation).

Table 1: Maximum-likelihood estimates for model parameters and associated life expectancies.

Year	$\hat{a}$	$\hat{b}$	$\hat{k}$	$[\hat{k}]$	$e_0(\hat{a}, \hat{b}, \hat{k})$	$e_0(\hat{a}, \hat{b}, [\hat{k}])$	$e_{30}(\hat{a}, \hat{b}, \hat{k})$	$e_{30}(\hat{a}, \hat{b}, [\hat{k}])$
1891	0.4488	0.10	10.26	10	73.12764	73.12776	42.20659	42.20671
1900	0.0418	0.11	8.64	9	75.24996	75.25373	44.27699	44.28077
1910	0.0370	0.11	8.97	9	76.08955	76.08812	45.11981	45.11837
1920	0.0374	0.11	9.17	9	75.93948	75.94496	44.97437	44.97984
1930	0.0379	0.10	9.69	10	75.34311	75.34210	44.39173	44.39073
1940	0.0388	0.11	8.75	9	75.80182	75.81748	44.82903	44.84469
1950	0.0340	0.12	8.23	8	77.28965	77.26412	46.30584	46.28031
1960	0.0287	0.13	7.91	8	78.80005	78.85181	47.81052	47.86228
1970	0.0218	0.13	7.80	8	81.02338	80.98194	50.00065	49.99920
1980	0.0180	0.13	7.66	8	82.37141	82.40531	51.39669	51.40060
1990	0.0147	0.13	7.62	8	83.70550	83.70612	52.71329	52.70737
2000	0.0119	0.14	7.40	7	85.10804	85.12865	54.10920	54.11260
2010	0.0091	0.14	7.12	7	86.53489	86.53414	55.49925	55.49785

Notes: Maximum-likelihood estimates $\hat{a}$ (at age 70), $\hat{b}$, $\hat{k}$ of the gamma-Gompertz parameters and respective life expectancies: $e_0(\hat{a}, \hat{b}, \hat{k})$ calculated by (2), $e_0(\hat{a}, \hat{b}, [\hat{k}])$ calculated by (4), $e_{30}(\hat{a}, \hat{b}, \hat{k})$ calculated by (10), and $e_{30}(\hat{a}, \hat{b}, [\hat{k}])$ calculated by a formula analogous to (4) (Data source: HMD (2012), Sweden, females; own estimation).

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